We can solve many problems concerning the motion of a particle in terms of the forces acting on it. The motion of a particle along a straight line (the x-axis) is described by its position function

\[ x = f(t) \]  

(1)

giving its x-coordinate at time \( t \). The velocity of the particle is defined to be

\[ v(t) = f'(t) \]  

(2)

\[ \frac{dx}{dt} \]

Its acceleration \( a(t) \) is

\[ a(t) = v'(t) = x''(t) \]

(3)

\[ \frac{dv}{dt} = \frac{d^2x}{dt^2} \]

Newton's second law of motion says that if a force \( F(t) \) acts on the particle and is directed along its line of motion, then

\[ m\ddot{x}(t) = F(t) \]  

(4)

\[ F = ma \]

Where \( m \) is the mass of the particle. If the force \( F \) is known, then the equation

\[ m\ddot{x} = \frac{d^2x}{dt^2} = \frac{F(t)}{m} \]

can be integrated twice to find the position function \( x(t) \) in terms of two constants of integration. These two arbitrary constants can be determined by the initial position \( x_0 = x(0) \) and the initial velocity \( v_0 = v(0) \) of the particle.
Case of constant acc. $a$: Suppose that the force $F$ and therefore the acc. $a = \frac{F}{m}$, are const. Then we begin with the expression:

$$\frac{dv}{dt} = a, \text{(where } a \text{ is const.)}$$

Integrating both sides we obtain:

$$v(t) = \int a \, dt = at + C_1.$$ 

Now, $v(0) = v_0 \implies v_0 = C_1$, \hspace{1cm} (5)

So, \hspace{1cm} $v(t) = \frac{dv}{dt} = at + v_0$ \hspace{1cm} (6)

Integrating again, we get:

$$x(t) = \int v(t) \, dt = \int (at + v_0) \, dt$$

$$= \frac{1}{2}at^2 + v_0 t + C_2$$

$x(0) = x_0 \implies x_0 = C_2$ \hspace{1cm} (7)

Therefore, \hspace{1cm} $x(t) = \frac{1}{2}at^2 + v_0 t + x_0$ \hspace{1cm} (8)

Thus, with equation (6) we can find the velocity and with equation (7) the position, of the particle at any time $t$ in terms of its constant acc. $a$, its initial velocity $v_0$ and its initial position $x_0$. 
Case of Vertical Motion with Gravitational Acc\(^2\);-

The weight \( W \) of a body is the force exerted on the body by gravity. Substituting \( a=g \) and \( F=W \) in Newton's second law \( F=ma \), we get

\[ W = mg \]  \( \text{(8)} \)

for the weight \( W \) of the mass \( m \) at the surface of the earth (where \( g \approx 32 \text{ ft/s}^2 \approx 9.8 \text{ m/s}^2 \)). For example, a mass of \( m=20 \text{ kg} \) has a weight of \( W = (20 \text{ kg})(9.8 \text{ m/s}^2) = 196 \text{ N} \).

To discuss vertical motion we choose the \( y \)-axis as the coordinate system for position, frequently with \( y=0 \) corresponding to the \text{ground level}. If we choose the upward direction as the \text{positive} direction, then the effect of gravity on a body vertically moving body is to decrease its \text{height} and also to \text{decrease} its \text{speed}. Consequently, if we ignore air resistance, \( v(t) = \frac{dy}{dt} \). Consequently, if we ignore air resistance, then the \text{acceleration} \( a = \frac{dv}{dt} \) of the body is given by

\[ \frac{dv}{dt} = -g \]  \( \text{(9)} \)

This \( a \) equation provides a starting point in many problems involving vertical motion. Successive integrations (as in Eqs (6) and (8)) gives the \text{velocity}

and \text{height} formulas

\[ v(t) = -gt + v_0 \]  \( \text{(10)} \)

\[ y(t) = -\frac{1}{2}gt^2 + v_0 t + y_0 \]  \( \text{(11)} \)

Here, \( y_0 \) denotes the \text{initial} \((t=0)\) \text{height} of the body and \( v_0 \) its \text{initial velocity}. 
Remark 1: If we neglect any effect of air resistance, then Newton's second law (\( F = ma \)) implies that the velocity \( v \) of the mass \( m \) satisfies the equation:

\[
\frac{mdv}{dt} = F_0
\]

where \( F_0 = -mg \) is the (downward-directed) force of gravity.

Remark 2: Next, if we want to take account of air resistance, the force \( F_R \) exerted by air resistance on the moving mass \( m \) must be added to eqn. (12), so we have:

\[
\frac{mdv}{dt} = F_0 + F_R
\]

Case of Resistance Proportional to Velocity:

We first consider the vertical motion of a body with mass \( m \) near the surface of the earth, subject to two forces:

(i) a gravitational (downward) force \( F_0 \) and

(ii) a force \( F_R \) of air resistance that is proportional to velocity, and is directed opposite the direction of the body.

If we set up a coordinate system with the positive \( y \)-direction upward and with \( y = 0 \) at ground level, then \( F_0 = -mg \) and \( F_R = -kv \)

where \( k \) is a positive constant and \( v = \frac{dy}{dt} \) is the velocity of the body.
6. Observation: The minus sign in Eqn. (14) makes $F_R$ positive (an upward force) if the body is falling ($v$ is $-ve$) and makes $F_R$ negative (a downward force) if the body is rising ($v$ is $+ve$).

As shown in the figure, the net force acting on the body is

$$F = F_R + F_G = -kv - mg.$$

So, using Newton's second law of motion $F = ma = m \frac{dv}{dt}$, we get

$$m \frac{dv}{dt} = -kv - mg$$

$$\Rightarrow \frac{dv}{dt} = -\frac{g}{k}v - g$$  \hspace{5cm} (15)

where $\frac{g}{k} > 0$.

Eqn. (15) is separable and can be written as

$$\frac{dv}{\frac{g}{k}v + g} = -dt$$

$$\Rightarrow \int \frac{dv}{\frac{g}{k}v + g} = -\int dt$$

$$\Rightarrow \ln |\frac{g}{k}v + g| = -st + C$$

$v(0) = v_0 \Rightarrow \ln |\frac{g}{k}v_0 + g| = C$

$$\Rightarrow \ln |\frac{g}{k}v + g| = -st + \ln |\frac{g}{k}v_0 + g|$$

$$\Rightarrow \ln \left(\frac{g}{k}v + g\right) = -st + \ln \left(\frac{g}{k}v_0 + g\right)$$

$$\Rightarrow \frac{g}{k}v + g = e^{-st} \left(\frac{g}{k}v_0 + g\right)$$

$$\Rightarrow v(t) = \left(\frac{g}{k}v_0 + g\right) e^{-st} - g$$

(16)

We note that $v_c = \lim_{t \to \infty} v(t) = -\frac{g}{k}$  \hspace{5cm} (17)
Thus the speed of a body falling with air resistance does not increase indefinitely. In fact, it approaches a finite limiting speed called terminal speed,

\[ V_T = \frac{g}{2} \Rightarrow \frac{V}{V_T} = \frac{2}{g} \]

Remark: The above fact makes a parachute a practical invention; it even helps explain the occasional survival of people who fall without parachutes from high-flying airplanes.

We now rewrite eqn. (16) in the form

\[ \frac{dy}{dt} = (V_0 - V_e) e^{-\alpha t} + V_e \]

\[ \Rightarrow \int dy = \int ((V_0 - V_e) e^{-\alpha t} + V_e) dt \]

\[ \Rightarrow y(t) = -\frac{1}{\alpha} (V_0 - V_e) e^{-\alpha t} + V_e t + C \]

\[ y(0) = y_0 \Rightarrow y_0 = -\frac{1}{\alpha} (V_0 - V_e) + C \]

\[ C = y_0 + \frac{1}{\alpha} (V_0 - V_e) \]

\[ y(t) = y_0 + \frac{1}{\alpha} (V_0 - V_e) - \frac{1}{\alpha} (V_0 - V_e) e^{-\alpha t} + V_e t \]

Eqs. (19) and (20) give the velocity \( v \) and height \( y \) of a body moving vertically under the influence of gravity and air resistance.
Example: The accel

Set: \[ \frac{dv}{dt} = k (250 - v) \]

\[ \Rightarrow \int \frac{dv}{250 - v} = \int k \, dt \]

\[ \Rightarrow -\ln |250 - v| = kt + C \]

\[ \Rightarrow \ln |250 - v| = -kt + C = -kt + \ln A \quad (C = \ln A) \]

\[ \Rightarrow 250 - v = A e^{-kt} \]

\[ \Rightarrow v(t) = A e^{-kt} + 250. \]

When \( t = 0 \), \( v = 0 \) (Given).

\[ \Rightarrow 0 = -A + 250 \Rightarrow A = 250 \]

\[ \therefore v(t) = -250 e^{-kt} + 250 \]

Again when \( t = 10 \), \( v = 100 \)

\[ \Rightarrow -100 = -250 e^{-10k} \]

\[ \Rightarrow 250 \cdot e^{-10k} = 100 \]

\[ \Rightarrow \frac{15}{25} = 0.6 = e^{-10k} \]

\[ \Rightarrow \ln (0.6) = -10k \]

\[ \Rightarrow 10k = 0.5108 \approx 0.511 \]

\[ \Rightarrow k = \frac{0.5108}{0.511} \approx 0.511 \]

\[ \therefore v(t) = -250 e^{-0.511t} + 250 \]

Suppose the car takes time \( T \) to accelerate to 200 km/h.

\[ \Rightarrow 200 = -250 e^{-0.511T} + 250 \]

\[ \Rightarrow 250 - 200 = 0.511T \]

\[ \Rightarrow 0.511T = 50 \]

\[ \Rightarrow T = \frac{50}{0.511} \approx 98 \text{ sec.} \]
\[ \frac{dv}{dt} = -kv \]

\[ \Rightarrow \frac{dv}{v} = -k \, dt \]

\[ \Rightarrow \ln v = -kt + \ln A \]

\[ v = A e^{-kt} \]

\[ v(0) = v_0 \Rightarrow v_0 = A e^{-k \cdot 0} \]

\[ v(t) = v_0 e^{-kt} \]

\[ \Rightarrow \frac{dx}{dt} = v_0 e^{-kt} \]

\[ \Rightarrow \int dx = \int v_0 e^{-kt} \, dt \]

\[ \Rightarrow x(t) = -\frac{v_0}{k} e^{-kt} + C \]

\[ x(0) = x_0 \Rightarrow x_0 = -\frac{v_0}{k} + C \]

\[ \Rightarrow C = x_0 + \frac{v_0}{k} \]

\[ x(t) = x_0 + \frac{v_0}{k} \left( 1 - e^{-kt} \right) \]

\[ \lim_{t \to \infty} x(t) = \lim_{t \to \infty} \left[ x_0 + \frac{v_0}{k} \left( 1 - e^{-kt} \right) \right] = x_0 + \frac{v_0}{k} \text{ which is finite.} \]

\[ \text{Sal} \]

\[ v(t) = \frac{dy}{dt} = \frac{dx}{dt} \]

\[ r(t) = \frac{dv}{dt} = -k \, v(t) \]

\[ \Rightarrow \frac{dv}{v} = -k \, dt \]

\[ \Rightarrow \ln v = -kt + \ln A \]

\[ v(t) = A e^{-kt} \]

\[ v(0) = \frac{dy}{dt} = \frac{dx}{dt} = \frac{d^2x}{dt^2} = v_0 \]

\[ v(t) = v_0 e^{-kt} \]

\[ x(t) = \frac{dx}{dt} = x_0 \]

\[ x(t) = x_0 + \frac{v_0}{k} \left( 1 - e^{-kt} \right) \]

\[ \lim_{t \to \infty} x(t) = x_0 + \frac{v_0}{k} \left( 1 - e^{-kt} \right) = \frac{40}{R} \left( 1 - e^{-kt} \right) = \frac{40}{R} \left( 1 - e^{-\frac{10}{ln2}} \right) = \frac{40}{R} = \frac{40}{\frac{1}{10} \ln 2} = 577.95 \text{ ft.} \]
2.3 Problems

1. The acceleration of a Maserati is proportional to the difference between 250 km/h and the velocity of this sports car. If this machine can accelerate from rest to 100 km/h in 10 s, how long will it take for the car to accelerate from rest to 200 km/h?

2. Suppose that a body moves through a resisting medium with resistance proportional to its velocity \( v \), so that \( dv/dt = -kv \). (a) Show that its velocity and position at time \( t \) are given by
   \[
   v(t) = v_0 e^{-kt}
   \]
   and
   \[
   x(t) = x_0 + \frac{v_0}{k} (1 - e^{-kt}).
   \]

   (b) Conclude that the body travels only a finite distance, and find that distance.

3. Suppose that a motorboat is moving at 40 ft/s when its motor suddenly quits, and that 10 s later the boat has slowed to 20 ft/s. Assume, as in Problem 2, that the resistance it encounters while coasting is proportional to its velocity. How far will the boat coast in all?

4. Consider a body that moves horizontally through a medium whose resistance is proportional to the square of the velocity \( v \), so that \( dv/dt = -kv^2 \). Show that
   \[
   v(t) = \frac{v_0}{1 + v_0 kt}
   \]
   and that
   \[
   x(t) = x_0 + \frac{1}{k} \ln(1 + v_0 kt).\]

   Note that, in contrast with the result of Problem 2, \( x(t) \to +\infty \) as \( t \to +\infty \).

5. Assuming resistance proportional to the square of the velocity (as in Problem 4), how far does the motorboat of Problem 3 coast in the first minute after it motor quits?

6. Assume that a body moving with velocity \( v \) encounters resistance of the form \( dv/dt = -kv^{3/2} \). Show that
   \[
   v(t) = \frac{4v_0}{(kt \sqrt{v_0} + 2)^2}
   \]
   and that
   \[
   x(t) = x_0 + \frac{2}{k} \sqrt{v_0} \left( 1 - \frac{2}{kt \sqrt{v_0} + 2} \right).
   \]

   Conclude that under a \( \frac{3}{2} \)-power resistance a body coasts only a finite distance before coming to a stop.

7. Suppose that a car starts from rest, its engine providing an acceleration of 10 ft/s², while air resistance provides 0.1 ft/s² of deceleration for each foot per second of the car’s velocity. (a) Find the car’s maximum possible (limiting) velocity. (b) Find how long it takes the car to attain 90% of its limiting velocity, and how far it travels while doing so.

8. Rework both parts of Problem 7, with the sole difference that the deceleration due to air resistance now is \((0.001)_x^2 ft/s^2\) when the car’s velocity is \( v \) feet per second.

9. A motorboat weighs 32,000 lb and its motor provides a thrust of 5000 lb. Assume that the water resistance is 100 pounds for each foot per second of the speed \( v \) of the boat. Then
   \[
   1000 \frac{dv}{dt} = 5000 - 100v.
   \]

   If the boat starts from rest, what is the maximum velocity that it can attain?

10. A woman bails out of an airplane at an altitude of 10,000 ft, falls freely for 20 s, then opens her parachute. How long will it take her to reach the ground? Assume linear air resistance \( pv \) ft/s², taking \( \rho = 0.15 \) without the parachute and \( \rho = 1.5 \) with the parachute. (Suggestion: First determine her height and velocity when the parachute opens.)

11. According to a newspaper account, a paratrooper survived a training jump from 1200 ft when his parachute failed to open but provided some resistance by flapping unopened in the wind. Allegedly he hit the ground at 100 mi/h after falling for 8 s. Test the accuracy of this account. (Suggestion: Find \( \rho \) in Eq. (4) by assuming a terminal velocity of 100 mi/h. Then calculate the time required to fall 1200 ft.)

12. It is proposed to dispose of nuclear wastes—in drums with weight \( W = 640 \) lb and volume 8 ft³—by dropping them into the ocean (\( v_0 = 0 \)). The force equation for a drum falling through water is
   \[
   m \frac{dv}{dt} = -W + B + F_R,
   \]
   where the buoyant force \( B \) is equal to the weight (at 62.5 lb/ft³) of the volume of water displaced by the drum (Archimedes’ principle) and \( F_R \) is the force of water resistance, found empirically to be 1 lb for each foot per second of the velocity of a drum. If the drums are likely to burst upon an impact of more than 75 ft/s, what is the maximum depth to which they can be dropped in the ocean without likelihood of bursting?

13. Separate variables in Eq. (12) and substitute \( u = v \sqrt{\rho/g} \) to obtain the upward-motion velocity function given in Eq. (13) with initial condition \( v(0) = v_0 \).

14. Integrate the velocity function in Eq. (13) to obtain the upward-motion position function given in Eq. (14) with initial condition \( y(0) = y_0 \).

15. Separate variables in Eq. (15) and substitute \( u = v \sqrt{\rho/g} \) to obtain the downward-motion velocity function given in Eq. (16) with initial condition \( v(0) = v_0 \).

16. Integrate the velocity function in Eq. (16) to obtain the downward-motion position function given in Eq. (17) with initial condition \( y(0) = y_0 \).