2.3 Problems

1. The acceleration of a Maserati is proportional to the difference between 250 km/h and the velocity of this sports car. If this machine can accelerate from rest to 100 km/h in 10 s, how long will it take for the car to accelerate from rest to 200 km/h?

2. Suppose that a body moves through a resisting medium with resistance proportional to its velocity \( v \), so that \( \frac{dv}{dt} = -kv \). (a) Show that its velocity and position at time \( t \) are given by

\[
v(t) = v_0 e^{-kt}
\]

and

\[
x(t) = x_0 + \left( \frac{v_0}{k} \right) (1 - e^{-kt}).
\]

(b) Conclude that the body travels only a finite distance, and find that distance.

3. Suppose that a motorboat is moving at 40 ft/s when its motor suddenly quits, and that 10 s later the boat has slowed to 20 ft/s. Assume, as in Problem 2, that the resistance it encounters while coasting is proportional to its velocity. How far will the boat coast in all?

4. Consider a body that moves horizontally through a medium whose resistance is proportional to the square of the velocity \( v \), so that \( \frac{dv}{dt} = -kv^2 \). Show that

\[
v(t) = \frac{v_0}{1 + v_0 k t}
\]

and that

\[
x(t) = x_0 + \frac{1}{k} \ln(1 + v_0 k t).
\]

Note that, in contrast with the result of Problem 2, \( x(t) \to +\infty \) as \( t \to +\infty \).

5. Assuming resistance proportional to the square of the velocity (as in Problem 4), how far does the motorboat of Problem 3 coast in the first minute after its motor quits?

6. Assume that a body moving with velocity \( v \) encounters resistance of the form \( \frac{dv}{dt} = -kv^{3/2} \). Show that

\[
v(t) = \frac{4v_0}{(kt \sqrt{v_0} + 2)^2}
\]

and that

\[
x(t) = x_0 + \frac{2}{k} \sqrt{v_0} \left( 1 - \frac{2}{kt \sqrt{v_0} + 2} \right).
\]

Conclude that under a \( \frac{3}{2} \)-power resistance a body coasts only a finite distance before coming to a stop.

7. Suppose that a car starts from rest, its engine providing an acceleration of 10 ft/s², while air resistance provides 0.1 ft/s² of deceleration for each foot per second of the car's velocity. (a) Find the car's maximum possible (limiting) velocity. (b) Find how long it takes the car to attain 90% of its limiting velocity, and how far it travels while doing so.

8. Rework both parts of Problem 7, with the sole difference that the deceleration due to air resistance now is \( 0.001 v^2 \) ft/s² when the car's velocity is \( v \) feet per second.

9. A motorboat weighs 32,000 lb and its motor provides a thrust of 5000 lb. Assume that the water resistance is 100 pounds for each foot per second of the speed \( v \) of the boat. Then

\[
1000 \frac{dv}{dt} = 5000 - 100v.
\]

If the boat starts from rest, what is the maximum velocity that it can attain?

10. A woman bails out of an airplane at an altitude of 10,000 ft, falls freely for 20 s, then opens her parachute. How long will it take her to reach the ground? Assume linear air resistance \( kv \) ft/s², taking \( k = 0.15 \) without the parachute and \( k = 1.5 \) with the parachute. (Suggestion: First determine her height and velocity when the parachute opens.)

11. According to a newspaper account, a paratrooper survived a training jump from 1200 ft when his parachute failed to open but provided some resistance by flapping unopened in the wind. Allegedly he hit the ground at 100 mi/h after falling for 8 s. Test the accuracy of this account. (Suggestion: Find \( k \) in Eq. (1) by assuming a terminal velocity of 100 mi/h. Then calculate the time required to fall 1200 ft.)

12. It is proposed to dispose of nuclear wastes—in drums with weight \( W = 640 \) lb and volume 8 ft³—by dropping them into the ocean \((v_0 = 0)\). The force equation for a drum falling through water is

\[
m \frac{dv}{dt} = -W + B + F_R,
\]

where the buoyant force \( B \) is equal to the weight (at 62.5 lb/ft³) of the volume of water displaced by the drum (Archimedes' principle) and \( F_R \) is the force of water resistance, found empirically to be 1 lb for each foot per second of the velocity of a drum. If the drums are likely to burst upon impact of more than 75 ft/s, what is the maximum depth to which they can be dropped in the ocean without likelihood of bursting?

13. Separate variables in Eq. (12) and substitute \( u = v \sqrt{\rho/g} \) to obtain the upward-motion velocity function given in Eq. (13) with initial condition \( v(0) = v_0 \).

14. Integrate the velocity function in Eq. (13) to obtain the upward-motion position function given in Eq. (14) with initial condition \( y(0) = y_0 \).

15. Separate variables in Eq. (15) and substitute \( u = v \sqrt{\rho/g} \) to obtain the downward-motion velocity function given in Eq. (16) with initial condition \( v(0) = v_0 \).

16. Integrate the velocity function in Eq. (16) to obtain the downward-motion position function given in Eq. (17) with initial condition \( y(0) = y_0 \).
\[
\frac{dv}{dt} = -kv^2 \\
\Rightarrow \frac{dv}{v^2} = -kdt \\
\Rightarrow \int \frac{dv}{v^2} = - \int kdt + C \Rightarrow \frac{-1}{v} = -kt + C \\
\text{Now } v(0) = v_0 \Rightarrow C = \frac{-1}{v_0} \\
\Rightarrow \frac{1}{v} = kt + \frac{1}{v_0} = \frac{k v_0 t + 1}{v_0} \\
\Rightarrow v = \frac{dy}{dt} = \frac{v_0}{1 + kv_0 t}
\]

\[
\Rightarrow dx = \frac{v_0}{1 + kv_0 t} dt
\]

Integrating again, we get
\[
x(t) = \frac{1}{k} \ln \left[1 + kv_0 t\right] + C
\]
\[
x(0) = x_0 \Rightarrow C = x_0
\]
\[
x(t) = \frac{1}{k} \ln \left(1 + kv_0 t\right) + x_0
\]
\[
\Rightarrow \lim_{t \to \infty} x(t) = \infty
\]

**Solution 5**

Using data of 83 in 84, as above, we get
\[
v = \frac{v_0}{1 + kv_0 t}
\]

Now \(v(10) = 20 \Rightarrow 20 = \frac{400}{1 + 400k} \Rightarrow 1 + 400k = 20 \Rightarrow k = \frac{1}{400}
\]

\[
\Rightarrow v(t) = \frac{400}{10 + t}
\]

\[
x(t) = 400 \ln \left(1 + \frac{t}{10}\right) = 400 \ln \left(\frac{10 + t}{10}\right)
\]

After 1 minute,
\[
x(60) = 400 \ln \left(\frac{10 + 60}{10}\right) = 400 \ln 7 \\
= 400 \ln 7 \\
= 677 \text{ ft}
\]
(a) As per the question,
\[ \frac{dv}{dt} = 10 - 0.1v \quad \Rightarrow \quad v(t) = \int (10 - 0.1v) \, dt = 10t - 0.1 \int v \, dt \]
\[ \frac{dv}{10 - 0.1v} = \frac{dt}{10} \quad \Rightarrow \quad \ln(10 - 0.1v) = -\frac{t}{10} + C \]
\[ v(0) = 0 \quad \Rightarrow \quad \ln 10 = C \]
\[ v(t) = 10e^{-t/10} \]
\[ \lim_{t \to \infty} v(t) = \lim_{t \to \infty} 100 \left(1 - e^{-t/10}\right) = 0 \]

(b) \[ \frac{dx}{dt} = v(t) = 100 \left(1 - e^{-t/10}\right) \]
\[ x(t) = \int 100 \left(1 - e^{-t/10}\right) \, dt = 100t + 1000 e^{-t/10} + C' \]
\[ x(0) = 0 \quad \Rightarrow \quad C' = -1000 \]
\[ x(t) = 100t + 1000 \left(1 - e^{-t/10}\right) \]

Suppose \[ v(t) = 90 \quad \Rightarrow \quad \int 100 \left(1 - e^{-t/10}\right) \, dt = 90 \quad \text{for} \quad t > T \]
\[ e^{-t/10} = 1 - \frac{90}{100} = \frac{1}{10} \]
\[ \Rightarrow \quad \frac{T}{10} = \ln \left(\frac{1}{10}\right) \quad \Rightarrow \quad T = 10 \ln 10 \approx 23 \text{ sec.} \]
\[ x(T) = 1402.59 \text{ ft} \approx 1403 \text{ ft.} \]
Here, \( V = 10 - 100t \), \( x(0) = V(0) = 0 \)

\[
\Rightarrow \int \frac{dv}{10 - 100t} = \int dt
\]

\[
\Rightarrow \frac{1}{100} \int \frac{dv}{1 - 100t^2} = \frac{1}{10} t + c
\]

\[
\Rightarrow \tan^{-1} \frac{V}{100} = \frac{t}{10} + c
\]

\( V(0) = 0 \Rightarrow 0 = c \)

\[
\Rightarrow \tan^{-1} \frac{V}{100} = \frac{t}{10}
\]

\[
\Rightarrow V(t) = 100 \tan^{-1} \left( \frac{t}{10} \right)
\]

\[
\lim_{t \to \infty} V(t) = \lim_{t \to \infty} 100 \tan^{-1} \left( \frac{t}{10} \right) = 100 \cdot \infty
\]

\[
\Rightarrow \lim_{t \to \infty} \frac{e^{t/10} - e^{-t/10}}{e^{t/10} + e^{-t/10}} = 100 \cdot \infty
\]

(b) \( \frac{dx}{dt} = V(t) = 100 \tan^{-1} \left( \frac{t}{10} \right) \)

\[
\Rightarrow dx = 100 \tan^{-1} \left( \frac{t}{10} \right) dt
\]

\[
\Rightarrow x(t) = \int 100 \tan^{-1} \left( \frac{t}{10} \right) dt
\]

\[
\Rightarrow x(t) = 1000 \ln(\cosh(\frac{t}{10})) + c'
\]

\( x(0) = 0 \Rightarrow c' = 0 \)

\[
\Rightarrow x(t) = 1000 \ln(\cosh(\frac{t}{10}))
\]

\[
\Rightarrow x(t) = 1000 \ln(\cosh(\frac{t}{10})) \quad \text{when} \quad t = T
\]

Suppose \( V(t) = 90 \) ft/sec \( \Rightarrow t = 14.722 \) sec.

\[
90 = 100 \tan^{-1} \left( \frac{t}{10} \right) \Rightarrow t = 14.722 \text{ sec}
\]

Then \( x(t) = 830.366 \) ft.